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Evaluating Active Fund Managers Using Time Series of Ex-ante Risk Estimates

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Abstract

Investors generally have less information about portfolios than their agent, the 'active fund manager', who actively manages funds entrusted by investors. Investors could reduce this informational disadvantage using time series of some ex-ante risk statistics pertaining to the portfolio. To achieve this, however, it is necessary to have reliable time series of risk estimates. In this article, we show the importance of ex-ante risk estimation in a multi-period setting, and propose a framework that enables the good use of time series of risk estimates to evaluate active fund managers.

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1. Introduction

In recent years, a new investment concept, founded on investment risk management, has been developed. Horie (2001) reports that some institutional investors in pension funds and the like are applying an approach based on this investment concept, which demonstrates investment decision-making as the allocation of a given amount of risk statistics. Because of this, the approach is dubbed 'risk allocation' or 'risk budgeting'.

One of the advantages of this approach is the systematic representation of multiple investment opportunities through a relatively small number of risk factors and their degree of influence (factor loading or exposure). Whenever an investor entrusts his or her investment to an external representative (called fund manager), a disparity emerges between them with respect to knowledge regarding individual investment opportunities or respective investment strategy. This knowledge disparity, referred to as 'asymmetric information', is thought to be an obstacle to efficient resource allocation (Kurasawa, 1988). Here, if an investor adopts an approach based on risk allocation, he (or she) can expect a reduction in informational disadvantage.

However, for the investor to actually enjoy the fruits of this advantage, it is necessary that the fund manager (who has access to richer information) estimates the risk structure of investment opportunities to some degree of accuracy. With respect to this point, many fund managers now have some equipment to measure their portfolio risks based on appropriate return models. Also, the emergence of such investors as mentioned above can be proof that risk estimation accuracy has reached a sufficient level.

In this paper, we demonstrate a framework that serves as a means to quantitatively evaluate the investment strategy of external active managers for investors who do not possess sufficient knowledge regarding individual investment opportunities or respective investment strategy. In the evaluation, the historical time series of ex-ante risk estimates of the evaluated strategy plays a very important role. In this paper, it is a presupposition that this time series would be obtained from the active manager, though investors themselves may just as well calculate it. The systematic evaluation shown herein can be utilized immediately by institutional investors such as pension funds[1] and the investment concept of investors (use of risk allocation approach or not) does not pose any problems at the primary level.

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[1] *Standard spreadsheet software would suffice. However, it is also possible to entrust this to an external consultant.*

The structure of this paper is as follows. The second section details the fundamental concept as background for active strategy evaluation. The third pans out the importance of risk management and the fourth sheds light on efficient risk management policy for active strategy (as a precondition of quantitative evaluation). The fifth section is the quantitative evaluation framework of active strategy based on management policy. The sixth is a conclusion.

2. Fundamental Concepts

2.1 Preparatory analysis of active strategy return

Suppose π_t is an active strategy portfolio over period t, which is a vector representing capital allocation to individual investment opportunities, then excess return r_t, which is univariate, for the strategy with respect to the riskless rate of interest for the period concerned is expressed by the following equation (where x' is the inversion of vector x):

$$\begin{aligned} r_t &= \pi_t' R_t \\ &= \pi_{M,t}' R_t + \pi_{A,t}' R_t \\ &= r_{M,t} + r_{A,t} \end{aligned} \qquad (1)$$

Here, $\pi_{M,t}$ is the benchmark portfolio and $\pi_{A,t}$ the portfolio structured at the discretion of the fund manager ($\pi_t=\pi_{M,t}+\pi_{A,t}$). Also, R_t is the aggregate of investment opportunities (excess returns from risk assets) over period t. The variables (π_t, $\pi_{M,t}$, $\pi_{A,t}$, R_t) are vectors; however, they may be considered as univariate for descriptive purposes.

In equation (1), it is shown that the excess return of active strategy can be divided into two terms. One is market-oriented variable $r_{M,t}$ which is not dependent on the discretion of the fund manager, and the other is manager-oriented variable $r_{A,t}$ which is dependent on his or her discretion. Below these, r_t can be called strategic return, $r_{M,t}$, market return, and $r_{A,t}$, active return.

If there are hedge opportunities against the fluctuation of market return $r_{M,t}$, active return $r_{A,t}$ becomes an independent investment opportunity and takes the name 'portable alpha'. If the time series of active return has no correlation with the series of market return $r_{M,t}$ (or, if the correlation is minimal)[2], it may be treated in the same order as the so-called hedge fund or alternative investments.

[2] *For example, in the case where fluctuation of the benchmark portfolio $\pi_{M,t}$ is moderate and the average position of the active portfolio $\pi_{A,t}$ would converge to zero over a given investment horizon ($(\pi_{A,1}+\cdots+\pi_{A,T})/T\approx 0$), the correlation of total returns of these two portfolio series could be ignored.*

Measuring active strategy risk using the tracking error (TE: ω_t) against the benchmark portfolio is consistent with the concept of perceiving active return $r_{A,t}$ ascribed to the manager's discretion as an independent investment opportunity.

$$\omega_t = \sqrt{\mathrm{var}_{t-1}(r_{A,t})} \qquad (2)$$

Here, $\mathrm{var}_t(\bullet)$ is a 'conditional variance operator' given all available information at the end of period t (we call such conditioning just 'conditioning' below). Also, under the concept above, both manager-oriented return $r_{A,t}$ and market-oriented return $r_{M,t}$ are considered excess returns with respect to the riskless rate of interest for the period. And, information ratio (IR: ϕ_t) pertaining to the former can therefore be immediately compared with the Sharpe ratio (SR: ψ_t) pertaining to the latter as a criterion in evaluating the efficiency of each investment opportunity (here, $\mathrm{E}_t(\bullet)$ is a 'conditional expectation operator').

$$\begin{aligned} \phi_t &= \alpha_t / \omega_t \\ \psi_t &= \mu_t / \sigma_t \end{aligned} \qquad (3)$$

$$\left(\alpha_t = \mathrm{E}_{t-1}(r_{A,t}), \quad \mu_t = \mathrm{E}_{t-1}(r_{M,t}), \quad \sigma_t = \sqrt{\mathrm{var}_{t-1}(r_{M,t})} \right)$$

2.2 Asymmetric information regarding active manager evaluation

This concept provides the very foundation for the approach that regards investment decisions as the allocation of some risk statistics. Investors adopting this approach take very seriously the expected TE and IR of an active strategy in manager selection. Why is this? It is because these parameters play a very important role when determining optimal risk allocation (including basic asset allocation, style management, passive/active ratio selection, and manager structure).

However, generally, optimal risk allocation is determined once in several years as investment policy, and the TE ($\overline{\omega}$) and IR ($\overline{\phi}$) used for decision-making purposes must be evaluated corresponding to a long-term or multi-period setting (t=1, 2, …, T)[3]. Equation (4) below shows that these two parameters are expressed using the short-term definitions of ω_t and ϕ_t over period t shown in equation (3).

$$\begin{aligned} \overline{\omega} &= \sqrt{\mathrm{var}_0(r_{A,1} + r_{A,2} + \cdots + r_{A,T})/T} \\ \overline{\phi} &= \{\mathrm{E}_0(r_{A,1} + r_{A,2} + \cdots + r_{A,T})/T\}/\overline{\omega} \end{aligned} \qquad (4)$$

[3] *Within this thesis, the long term and short term are abstract periods. However, to have a more concrete image, one may consider the former as representing three to five years and the latter, one month.*

Investors pay particular attention to: (1) whether expected TE can be realized in the long term and (2) whether expected IR can be achieved in the same term. However, these parameters cannot be observed and are difficult to evaluate with any precision. Because of this, these two points are commonly evaluated by employing a statistical method based on an observed sample (time series of realized returns).

One of the methods often used is to postulate that short-term active return $r_{A,t}$ is distributed independently and identically over time (referred to as i.i.d. assumption on $r_{A,t}$), and estimate the long-term TE $\overline{\omega}$ and IR $\overline{\phi}$ using observed time series of the short-term active return $\dot{r}_{A,t}$ in the past ($t=\tau+1,\ \tau+2,\cdots,\ \tau+N\leq 0$) based on the formula (equation (5))[4] below. Nevertheless, these estimated values have some estimation error, even if model selection (in this case, the i.i.d. assumption on the active return) is valid.

$$\dot{\overline{\omega}} = \sqrt{\{\textstyle\sum_{t=\tau+1}^{\tau+N} \dot{r}_{A,t}^2 - N(\sum_{t=\tau+1}^{\tau+N} \dot{r}_{A,t} / N)^2\}/(N-1)}$$
$$\dot{\overline{\phi}} = (\textstyle\sum_{t=\tau+1}^{\tau+N} \dot{r}_{A,t} / N) / \dot{\overline{\omega}} \qquad (5)$$

The advantage of this method is that it is possible to make an easy estimate of the required parameters without any special knowledge if a time series sample of active return $\dot{r}_{A,t}$ is available. In fact, it seems to be used by many investors to evaluate active strategies.

However, this method can involve a significant problem because the selected active return model (i.i.d. assumption) is often improper. The problem rises to the surface when the active portfolio is rebalanced ($\pi_{A,t}\neq\pi_{A,t-1}$).

$$\omega_t = \sqrt{\pi'_{A,t}\Omega_t\pi_{A,t}} \qquad (6)$$

Equation (6) is a parametric expression of risk (TE: ω_t) for an active strategy over period t. Here, Ω_t represents a conditional covariance matrix of aggregate investment opportunities R_t ($\Omega_t=\mathrm{var}_{t-1}(R_t)$; this also may be considered a univariate). As can be seen from the equation, even if the i.i.d. assumption for a set of investment opportunities R_t is true and $\Omega_t=\Omega$ ($t=1, 2,\cdots, T$) is supported, then there is no guarantee that return $r_{A,t}$ from the active strategy with some rebalancing ($\pi_{A,t}\neq\pi_{A,t-1}$) conforms to i.i.d.

[4] *Qualitative analysis is often applied in IR evaluation which requires estimation of expected excess return of an active strategy. The manager evaluation framework demonstrated in section 5 of this*

As iterated in the following, with regard to active strategy with some rebalancing, dynamic TE (ω_t) control significantly affects the strategy's efficiency and return properties. Hence, the ex-ante estimation of TE is an essential factor for the strategy. As shown in equation (6), the TE of the active strategy over period t can be calculated if the conditional covariance matrix Ω_t of investment opportunity and active portfolio $\pi_{A,t}$ for the period concerned are known. A fund manager executing investment strategy would be expected to estimate these parameters to an appropriate degree. On the other hand, client investors who can measure these parameters for themselves are in the minority; moreover, those who actually exploit them are even smaller in number.

The asymmetry of this information can be dissolved merely by asking each fund manager, who is the agent or candidate, for the information concerned (ω_t; $t=\tau+1, \tau+2,\cdots, \tau+N$). The following will clarify the role of dynamic risk management in active strategy and show ways to use two types of historical time series, ex-ante risk estimates and ex-post realized returns, to quantitatively evaluate the strategy.

3. Importance of Dynamic Risk (TE) Management of Active Strategy

It is important to understand the role of dynamic measurement of active strategy risk (TE) from two points of view. First, whether expected TE is realized entirely over the evaluation term (long term consisting of shorter periods), and second whether it is realized effectively over the same periods. The former focuses attention on the total volume management of TE over the evaluation periods (considered as the scale aspect of risk control) and the latter on TE allocation management within the overall evaluation periods (considered as the skill aspect of risk control). Below is a simple explanation of the importance of these two points.

3.1 Importance of managing total amount of active risk (TE)

Generally speaking, risk management often refers to reducing or eliminating risk (for example, the case of operation risk, etc.). However, investors expect the active manager to take active risk, so any unexpected avoidance of risk taking for the long term would be contrary to the intention of investors. There are two aspects to this: (1) a direct aspect where the expected active return (alpha) decreases and (2) an indirect one where investment opportunities available to investors are wasted.

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thesis is applicable irrespective of the IR estimation method and expected excess return.

Aspect (1) can be easily understood by recalling the IR definitional equation. In other words, even if IR $\overline{\phi}$ is as expected, if TE $\overline{\varpi}$ does not reach the expected level, alpha $\overline{\alpha}$ (= $E_0(r_{A,1}+r_{A,2}+\cdots+r_{A,T})/T$) will not be achieved in the long term.

$$\overline{\alpha} = \overline{\phi} \times \overline{\varpi} \qquad (7)$$

In addition, even if IR $\overline{\phi}$ exceeds an expected level ex post facto and the target alpha is well achieved, there remains a problem of wasting a part of the investor's risk budget in vain. In other words, suppressing TE contrary to the investor's initial intention is the same as stealing additional investment opportunities that should have been taken head on by the investor. This is aspect (2) above.

As such, the active manager should control active risk level with a keen eye on the objective and not just keep risk below the upper limit.

3.2 Importance of managing the allocation of active risk (TE)

Another point with respect to TE management is 'effectively realizing the expected TE over the evaluation periods. This means that whenever an active investment opportunity can be divided into sub-opportunities, the active manager is required to allocate the total risk budget efficiently to maximize the IR of her or his strategy over these periods.

The division of investment opportunity is twofold, cross-sectional focusing on the sources of the risk, and dynamic (or chronological) focusing on the change in investment opportunity. The former is equivalent to ordinary risk budgeting within each strategy. This cross-sectional allocation of risk amount should be executed within the active strategies that contain multiple sources of excess return (for example, 'strategy focusing on (1) sector rotation and (2) stock selection', etc.).

Contrastingly, the division of investment opportunity within periods is common for any active strategies with portfolio rebalancing. In particular, managers who provide some kind of active strategies, such as tactical asset allocation or active duration control, would take it seriously. In these strategies, exposure to concerned risk factors should be deliberately changed, then active risk levels would fluctuate over the periods. Also, the same exists with strategies where the fund manager controls risk exposure according to the degree of certainty of his or her view.

Considering the efficiency of active strategies, these two risk allocations both play an important role. However, in many cases, 'risk allocation' often refers to cross-sectional risk allocation, rarely to dynamic risk allocation[5]. Because of this, in the following we shed light on the matter and organize a concept in a framework focusing on dynamic risk allocation (the same theory could be formulated for cross-sectional risk allocation).

3.3 Importance of dynamic risk allocation

Consider two active strategies with an investment horizon of one year. One of them is a strategy with a constant TE of 1% every month ($\omega_1=\omega_2\cdots=\omega_{12}=1\%$) and the other is a strategy where $\sqrt{12}$ % of the TE is taken for only the last month ($\omega_1=\omega_2\cdots=\omega_{11}=0\%$, $\omega_{12}=\sqrt{12}$ %). Looking only at the TE $\bar{\omega}$ over all investment periods, both of these strategies are the same. However, if the short-term IR is constant ($\phi_1=\phi_2\cdots=\phi_{12}=\phi$) over these periods, the strategy with the constant TE (1% every month) has the larger IR $\bar{\phi}$ over the entire year.

$$\frac{12(\phi\times 1\%)}{\sqrt{12}\%} > \frac{11(\phi\times 0\%)+1(\phi\times\sqrt{12}\%)}{\sqrt{12}\%}$$

The above inequality is a comparison of the long-term IR of these two strategies over the entire year. The left side of the inequality (=$\sqrt{12}\,\phi$) represents the full-term IR of the uniform TE strategy, and arrives at $\sqrt{12}$ times the right side (=ϕ) which represents the non-uniform TE. This shows that the manager who employed the latter strategy did not make good use of the total risk budget (or given investment opportunities).

However, this does not mean that the latter strategy is always inefficient. For example, suppose a fund manager has a better forecast about a rare event that affects asset return at the corresponding period (here, the 12th month). And suppose the IR of his strategy over this period is heightened ten times the previous months' ($\phi_1=\phi_2\cdots=\phi_{11}=\phi$, $\phi_{12}=10\phi$), the latter non-uniform TE strategy would have a higher IR $\bar{\phi}$ over the entire year. In the inequality below, the full-period IRs of the two strategies above are compared. The long-term IR of the strategy with constant TE (the left side=$21\phi/\sqrt{12}$), is only about 0.6 times the IR of the non-uniform TE strategy (right side=10ϕ).

$$\frac{11(\phi\times 1\%)+1(10\phi\times 1\%)}{\sqrt{12}\%} < \frac{11(\phi\times 0\%)+1(10\phi\times\sqrt{12}\%)}{\sqrt{12}\%}$$

In other words, the active manager, depending on his own evaluation of the short-term IR ϕ_t,

[5] *Grinold/Kahn (1995), referred to by many practitioners, makes little mention of the dynamic management of risk.*

could increase the long-term IR $\overline{\phi}$ over the evaluation periods by controlling the short-term TE level ω_t appropriately. And, doing so, he could implement the good use of a given amount of risk which is a limited resource of the investor.

4. Efficient Risk Management of Active Strategy

4.1 Efficient TE management strategy for fund managers

Given a time series of short-term IR over straight T periods (ϕ_1, ϕ_2,･･･, ϕ_T) and overall TE level $\overline{\omega}$ over these periods (t =1, 2,･･･, T), the short-term TE control for the fund manager's optimal strategy becomes proportional to the IR level over the concerned period. Below is a simple explanation of this point.

First, consider the long-term IR $\overline{\phi}$ over T periods. The expected excess return 'alpha' over period t is expressed as $\alpha_t = \phi_t \times \omega_t$ (t=1, 2,･･･, T); here ϕ_t and ω_t are short-term IR and TE over period t respectively. IR ϕ_t, corresponding to the forecasting capability of an active manager over period t, is given, and TE ω_t is a control variable for the manager. Under these settings, long-term expected excess return $\overline{\alpha}$, TE $\overline{\omega}$, and IR $\overline{\phi}$ over the T periods are shown in the equation below.

$$\begin{aligned} \overline{\alpha} &= \textstyle\sum_{t=1}^{T} \alpha_t / T \\ \overline{\omega} &= \sqrt{\textstyle\sum_{t=1}^{T} \omega_t^2 / T} \\ \overline{\phi} &= \frac{\sum_{t=1}^{T} (\phi_t \times \omega_t)}{\sqrt{T \sum_{t=1}^{T} \omega_t^2}} \end{aligned} \qquad (8)$$

Furthermore, the short-term TE level ω_t satisfies the following first-order condition for optimal TE management.

$$\frac{\partial \overline{\phi}}{\partial \omega_t} = \frac{\phi_t}{\sqrt{T \sum_{t=1}^{T} \omega_t^2}} - \frac{\sum_{t=1}^{T} (\phi_t \times \omega_t)}{\sqrt{T}\left(\sqrt{\sum_{t=1}^{T} \omega_t^2}\right)^3} \omega_t = 0$$

Solving this first-order condition given a total TE level $\overline{\omega}$ leads us to the following necessary condition about the short-term TE level ω_t under optimal TE management.

$$\omega_t = k \times \phi_t \qquad (9)$$

Here, multiplier k is a constant coefficient over the periods given by

$k = [\sum_{t=1}^{T} \omega_t^2] \big/ [\sum_{t=1}^{T} (\phi_t \times \omega_t)] = \overline{\omega} / \overline{\phi} > 0$. And, it must be noted that within the short-term IR series (ϕ_1, ϕ_2,・・・, ϕ_T), aside from the first term ϕ_1 , the rest are generally uncertain at investment start-up t=0 (see equation (4)). Because of this, to determine multiplier k, it becomes necessary to clarify the joint distribution of the short-term IRs of period 2 and beyond (ϕ_2, ϕ_3,・・・, ϕ_T). In this paper, to simplify problems, as mentioned at the beginning of this subsection, it is postulated that the whole time series of the short-term IR even beyond period 1 (ϕ_1, ϕ_2,・・・, ϕ_T) are known at the beginning of investment (t=0).

4.2 Maximization of investor expected utility function and active IR strategy

To simplify things, the non-correlation between market return $r_{M,t}$ and active return $r_{A,t}$ is assumed. In addition, the investor considers using both passive and active managers, and his or her expected utility function $\bar{U}$ over investment periods (t=1, 2,・・・, T) is given in the following equation.

$$\begin{aligned} \overline{U} &= \mathrm{E}_0[\textstyle\sum_{t=1}^{T} U_t] \\ U_t &= \mathrm{E}_{t-1}[\eta_t r_{M,t} + \theta_t (r_{M,t} + r_{A,t})] - 0.5\lambda \,\mathrm{var}_{t-1}[\eta_t r_{M,t} + \theta_t (r_{M,t} + r_{A,t})] \\ &= [(\eta_t + \theta_t)\mu_t + \theta_t \alpha_t] - 0.5\lambda[(\eta_t + \theta_t)^2 \sigma_t^2 + \theta_t^2 \omega_t^2] \end{aligned} \tag{10}$$

Here, U_t is the investor short-term expected utility function over period t and λ represents the investor risk aversion level. Also, μ_t, α_t, σ_t, and ω_t are conditional expected values (μ_t, α_t) and conditional standard deviations (σ_t, ω_t) for market return and active return over period t, respectively. η_t and θ_t represent capital allocation to passive and active managers over period t, respectively. Among these parameters, ω_t is the active manager control variable and η_t and θ_t are investor control variables.

Further, as in subsection 4.1, in order to simplify matters, the short-term IR series (ϕ_2, ϕ_3,・・・, ϕ_T) relating to the active strategy and the series of conditional expectations and standard deviation (μ_2, μ_3,・・・, μ_T and σ_2, σ_3,・・・, σ_T) relating to the passive strategy are assumed to be given at the start of period 1 (t=0) deterministically. At this time, an investor maximizing his expected utility function $\bar{U}$ should satisfy the simultaneous first-order condition below.

$$\frac{\partial \overline{U}}{\partial \omega_t} = \theta_t(\phi_t - \lambda\theta_t\omega_t) = 0$$
$$\frac{\partial \overline{U}}{\partial \eta_t} = \mu_t - \lambda(\eta_t + \theta_t)\sigma_t^2 = 0$$
$$\frac{\partial \overline{U}}{\partial \theta_t} = [\mu_t - \lambda(\eta_t + \theta_t)\sigma_t^2] + \omega_t(\phi_t - \lambda\theta_t\omega_t) = 0$$

To obtain the above equations, the relation between the conditional expected excess return of active strategy α_t and the short-term information ratio ϕ_t for the current period is employed: $\alpha_t = \phi_t \times \omega_t$. Solving the above equations, as long as equation (9) is satisfied through the active manager's own strategy, the investor only has to keep the capital allocation ratio θ_t to the active manager constant (θ^*) in order to maximize his own expected utility $\bar{U}$ (see equation (11)).

$$\begin{aligned}\theta^* &= \phi_t/(\lambda\omega_t) \\ &= 1/(\lambda k) \\ &= \bar{\phi}/(\lambda\bar{\varpi})\end{aligned} \qquad (11)$$

In other words, investors who want to employ a static policy for capital allocation to active managers should make them maximize their own long-term IR by adopting the dynamic TE control rule shown by equation (9)[6].

In addition, the optimal amount (or total budget) of active risk over T periods for an investor, $\sqrt{T}\bar{\varpi}^*$, would be expressed in equation (12) by combining the definition of multiplier k ($=\bar{\varpi}/\bar{\phi}$), shown in subsection 4.1, with the long-term IR maximization condition ($\omega_t = k \times \phi_t$) and also the condition to maximize the investor's expected utility function shown in equation (11).

$$\begin{aligned}T(\bar{\varpi}^*)^2 &= \textstyle\sum_{t=1}^{T}(\theta^*\omega_t)^2 \\ &= (\theta^* k)^2 \textstyle\sum_{t=1}^{T}\phi_t^2 \\ &= (1/\lambda)^2(\textstyle\sum_{t=1}^{T}\phi_t^2)\end{aligned} \qquad (12)$$

From this equation, it is understood that the total amount of active risk that should be taken by the investor decreases according to the degree of his or her risk aversion and increases according to the long-term IR level of the active strategy. Further, for the investor, the long-term IR level $\bar{\phi}^*$ over the investment periods concerned is given in the following equation, which shows that it is not dependent on investor attributes.

$$\bar{\phi}^* = \frac{\theta^*\sum_{t=1}^{T}\alpha_t/T}{\bar{\varpi}^*} = \frac{\theta^*\sum_{t=1}^{T}(\phi_t \times \omega_t)}{(1/\lambda)\sqrt{T\sum_{t=1}^{T}\phi_t^2}} = \frac{\sum_{t=1}^{T}(\phi_t^2/\lambda)}{(1/\lambda)\sqrt{T\sum_{t=1}^{T}\phi_t^2}} = \sqrt{\textstyle\sum_{t=1}^{T}\phi_t^2/T}$$

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[6] *If μ_t/σ_t^2 is not constant, investors need to allocate capital to passive managers dynamically ($\eta_t^* = \mu_t/(\lambda\sigma_t^2) - \theta^*$) in order to maximize their own expected utility functions. However, if passive managers control investment proportion in safe assets and market index vehicles based on this rule (including the TAA of overlay manager), then optimal capital allocation to passive managers also becomes constant.*

5. Method of Active Strategy Evaluation

Consequent to the organization shown in the previous sections, we will demonstrate a framework of quantitative evaluation of active strategy using a time series of ex-ante TE estimates. This evaluation consists of five points below (for investors who make their own TE estimate, the first point evaluates their own TE estimation capability).

5.1 Validity of short-term TE estimation

The first point sheds light on the validity of an ex-ante estimation of short-term TE. This estimation is equivalent to the infrastructure of the active strategy. The active manager is expected to estimate the ex-ante TE for his own strategy dynamically in an effort to keep it efficient (see equation (9)). To the contrary, if the TE estimate is inaccurate, the efficiency of the strategy cannot be guaranteed. Also, it is a precondition for the evaluation framework shown below that the short-term TE is accurately estimated.

Here, two evaluation methods are shown for the validity of ex-ante TE estimates over given N periods (t=τ+1, τ+2,・・・, τ+N) based on the concerned active returns ($\dot{r}_{A,\tau+1}, \dot{r}_{A,\tau+2}, \cdots, \dot{r}_{A,\tau+N}$) over the same periods. One of the methods compares two estimates for the true long-term TE $\overline{\varpi}$ expressed in equation (8). One estimate, $\hat{\overline{\varpi}}$, is shown by the equation below, which substitutes the ex-ante TE estimates ($\hat{\omega}_{\tau+1}, \hat{\omega}_{\tau+2}, \cdots, \hat{\omega}_{\tau+N}$) for true TE series in equation (8). Another estimate is a sample standard deviation $\dot{\overline{\varpi}}$ obtained from equation (5) using ex-post (realized) active returns.

$$\hat{\overline{\varpi}} = \sqrt{\sum_{t=\tau+1}^{\tau+N} \hat{\omega}_t^2 / N}$$

As long as short-term TE ω_t and IR ϕ_t are not constant over periods, the sample variance $\dot{\overline{\varpi}}^2$ based on equation (5) should overestimate the squared long-term TE $\overline{\varpi}^2$ in equation (8). Details have been omitted[7]. However, this bias is equivalent to the sample variance of α_t=$\phi_t\,\omega_t$ (t=τ+1, τ+2, $\cdots$, τ+N) and is correctable. Also, ordinarily, this bias can be ignored because it is considered relatively minimal (with respect to $\overline{\varpi}^2$).

The problem with this method is that available information about the correspondence of

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[7] *Readers who desire details pertaining to this part, or footnote 8, are asked to contact the author (nakashima@mizuho-pri.co.jp).*

ex-post return $\dot{r}_{A,t}$ to ex-ante risk $\hat{\omega}_t$ over each period is not exploited. In other words, even if the total amount of estimated TE is consistent with realized returns over the sample periods, it is not yet shown that a respective TE estimate over each period is correct. To avoid this problem, it is effective to exploit the time series of standardized return, which is calculated by dividing observed active return by the ex-ante TE estimate concerned at each period ($\dot{r}_{A,t} / \hat{\omega}_t$).

Under the hypothesis that TE is properly estimated, this standardized return is considered a sample from the probability distribution with mean ϕ_t and variance one. Further, if this probability distribution is normal, and if the fluctuation of the mean level ϕ_t over periods is so small that the sample variance of ϕ_t is negligible compared to one (actually, the slope of the solid line is sufficiently small in Figure 1 below), then the product of the sample variance of standardized returns multiplied by sample size N follows the chi-square distribution with approximately N-1 degrees of freedom[8].

In addition, should the ex-ante TE estimates be correct, the sample kurtosis of standardized returns is on average smaller than the sample kurtosis of active returns[9]. Contrastingly, if TE estimates are not accurate, the sample variance of standardized returns on average overestimates true variance (equal to one) according to Jensen's inequality.

As iterated above, should the TE estimation errors cancel out each other over the sampling periods, they cannot be detected using the first method mentioned. However, with the method that utilizes standardized return, by focusing attention on the abovementioned two points, the validity of ex-ante TE estimates can be confirmed.

5.2 Implementation of investment with entrusted TE level

The second evaluation point is to question whether total TE level over the evaluation periods was appropriate. As mentioned, over the investment periods, the active strategy manager must use up entrusted risk (from the investor) at the commencement of investment. As such, this evaluation point uses no information about ex-post returns, so it is referred to as an evaluation from a scale

[8] *If the conditional mean of IR ϕ_t over period t fluctuates considerably, then this approach underestimates both the true mean and variance of standardized return. Consequently, the test using this approach would be conservative for the overestimation bias contained in the following sample variance. This means that power is enhanced in the test of accuracy of TE estimation.*

[9] *This inequality holds for 'expectation' of sample kurtosis, but not always for respective sample kurtosis. Readers desiring details pertaining to this theory are asked to contact the author (see footnote 7).*

perspective. This evaluation point will not be problematic as long as short-term TE is accurately estimated (see 5.1 above).

5.3 Achieving expected IR in the long term

The third point is whether expected IR is achieved in the long term ex post facto. Because this point includes an evaluation of expected excess return, it is referred to as evaluation from a skill perspective. However, the *mean estimate* error is relatively large compared to *variance estimate* error. Because of this, situations that can lead to a definite statement about the investment skill of an active manager are limited. In other words, we cannot often reject either the null hypothesis based on standard statistics that 'the manager has no investment skill ($\bar{\phi}$ =0)' or the null hypothesis that 'the manager has expected skills $\phi^{\#}$ ($\bar{\phi}=\phi^{\#}$)'. This means that we can rarely know whether the manager has true investment skill or not.

Incidentally, if the active return follows $r_{A,t}=\alpha_t+\omega_t\varepsilon$ (ε_t～i.i.d.N(0,1)), the following relation holds (see equation (4)).

$$\begin{aligned}\textstyle\sum_{t=\tau+1}^{\tau+N}(\dot{r}_{A,t}/N)/\overline{\varpi} &= \textstyle\sum_{t=\tau+1}^{\tau+N}(\alpha_t+\omega_t\dot{\varepsilon}_t)/(N\overline{\varpi}) \\ &= \bar{\phi}+\textstyle\sum_{t=\tau+1}^{\tau+N}\omega_t\dot{\varepsilon}_t/(N\overline{\varpi})\end{aligned} \qquad (13)$$

Here, because of equation $N\overline{\varpi}^2=\sum_{t=\tau+1}^{\tau+N}\omega_t^2$, the second term on the right side of equation (13) follows normal distribution N(0,1/*N*). Using this relation, a null hypothesis $\bar{\phi}=\phi$' can be tested for any level of expected investment skill ϕ'.

5.4 TE allocation to prescribed risk sources

The fourth evaluation point is whether cross-sectional TE allocation is appropriate for the active strategy concerned, which has multiple risk factors as sources of excess return.

The efficient TE management principle shown in equation (9) is not only valid chronologically but also cross-sectionally. In the latter case, subscript *t* representing the time period in this equation should be replaced with another subscript, for instance *j* (=1, 2, ⋯, *J*), which represents respective source of risk (risk factor) for excess return. As such, the optimal cross-sectional TE allocation should be implemented in proportion to the IR level of respective risk source of the strategy, as long as there is no frictional problem caused by liquidity restrictions, etc. In other words, a manager who non-uniformly allocates total TE to prescribed sources of risk is implicitly exhibiting a disparity in his own forecasting capability among these factors.

As a result, by directly comparing average TE allocation over the investment periods estimated at the end of the periods with the prescribed allocation at the start of the periods, the consistency of self-knowledge of the fund manager can be evaluated from a kind of scale perspective. In this regard, if the investor allows the fund manager to correct his or her self-knowledge with respect to capability to forecast respective risk factors, the test shown in subsection 5.3 could be carried out for each factor. However, as already mentioned, it is considered that cases where clear results can be obtained using the latter verification are limited.

5.5 Efficiency of dynamic TE control

It is thought to be difficult to effect an evaluation relating to a fund manager's skill with respect to the average of long-term IR. However, it is possible to extract a comparatively clear conclusion about his or her skill if we implement an evaluation with respect to the efficiency of dynamic TE control under a given risk budget.

The fifth evaluation point is determining whether the given amount of TE, corresponding to the total risk budget over investment periods, is efficiently used. As iterated in subsection 4.1, if an active manager is maximizing long-term IR of his or her strategy, the relation between short-term IR and TE shown in equation (9) holds; $\omega_t = k \times \phi_t$.

Then, standardized return $r_{A,t}/\omega_t$, active return over period t divided by TE over the period, is distributed around current short-term IR ϕ_t under the assumption mentioned in subsection 5.3 ($r_{A,t}=\alpha_t+\omega_t\varepsilon_t$, $\varepsilon_t \sim$ i.i.d.N(0,1)).

$$r_{A,t} / \omega_t = \phi_t + \varepsilon_t = (1/k)\omega_t + \varepsilon_t$$

Consequently, if the active strategy is efficiently managed, a tendency for a proportional relation should be observed between standardized return $r_{A,t}/\omega_t$ and current TE ω_t in a scatter diagram where the combinations of these two variables are plotted over evaluated periods. Also in this diagram, the horizontal axis, meaning some cause rather than effect, should represent current TE as a control variable. In other words, by testing the inclination of the regression line in this scatter diagram, it is possible to verify whether TE is efficiently managed from the perspective of dynamic control.

Standardized return $r_{A,t}/\omega_t$ is equivalent to the return in an alternative strategy where the active portfolio $\pi_{stdA,t}$ is obtained by multiplying active portfolio $\pi_{A,t}$ in the original strategy by a scalar c_t. Here, the scalar coefficient is expressed as $c_t=1/\omega_t$, so this alternative strategy with portfolio $\pi_{stdA,t}$ should be called something like 'constant TE strategy'. It goes without saying that the TE for this alternative strategy is always one. Conversely, in order for this alternative strategy to be efficient, a uniform short-term IR is necessary ($\phi_1=\phi_2=\cdots=\phi_T=\phi$; see equation (9)).

Because of this, the null hypothesis that an active manager's short-term IR is uniform can be tested, by comparing the IR of the constant TE strategy over the evaluation periods with the IR of the original strategy to be evaluated. Effecting this validation as a preliminary to the validation using the abovementioned scatter diagram, the overall perspective improves.

Also, aside from this validation, it is important to view the chart plotting the time series of ex-ante estimated TE. If this chart contains seasonality, a trend, or rapid changes, moral hazard (negligence and improper risk taking, etc.) on the part of the fund manager should be suspected[10].

5.6 Case study: evaluating dynamic TE control

Figure 1 shows a case which evaluates dynamic TE management for a real active fund in the way introduced in subsection 5.5. Each dot plotted in this figure represents a combination of ex-ante TE estimate over month t ($\hat{\omega}_t$; measured by the horizontal axis) and corresponding ex-post standardized return ($\dot{r}_{A,t}/\hat{\omega}_t$; measured by the vertical axis). Though each variable within the combination is a monthly sampling, the scale displayed is suitably annualized.

In Figure 1, the solid line represents efficient TE management for this strategy. So it should have an uptrend because it is equivalent to equation (9). Further, under the assumption for the standardized return over each month to follow normal distribution, standardized return is dotted between two broken lines in the same figure with 95% probability. In actuality, looking at the figure, 93% of the entire sample falls within the same area so the efficiency of the strategy seems to be supported.

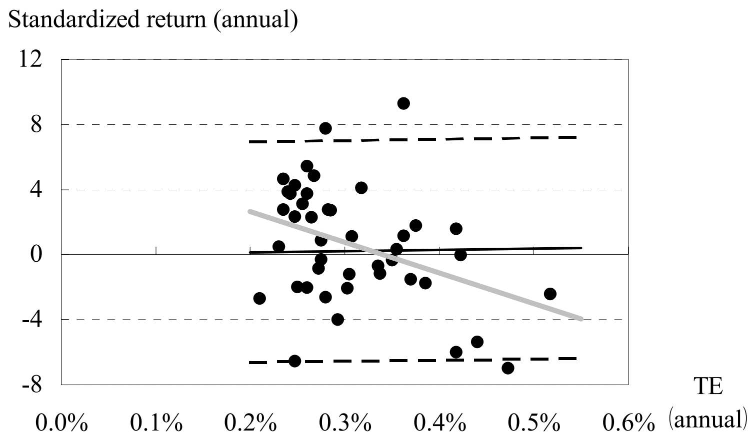
However, from this point of view, the gray line in this figure plays the critical role. It is obtained by regressing standardized return to the TE estimate concerned. In Figure 1, this regression line has such an obvious downtrend that a hypothesis that TE is efficiently controlled over the periods is significantly rejected by standard statistics (the t value of the regression

[10] *Yano (2001) introduces some studies analyzing such moral hazard.*

coefficient becomes -2.6). This conclusion holds for either strategy whether short-term IR is fluctuating or not (for the former, efficient control is expressed in the uptrend line, for the latter, the flat line). The regression line shown in Figure 1 implies that a confident forecast made by an active manager might have negative bias in a multi-period setting.

Actually, the realized long-term IR of the active strategy studied in Figure 1 has reached 0.25 annually. However, just by flattening TE level over the periods in a simplified manner ('constant TE strategy'), the long-term IR over the same periods improved to 0.57 annually. In other words, the fund manager responsible for this type of strategy may very well have some kind of perception bias (systematic error relating to the short-term IR estimate).

Figure 1: Relation between the ex-ante TE estimate $\hat{\omega}_t$ and standardized return $\dot{r}_{A,t} / \hat{\omega}_t$ for a given strategy (monthly sample data displayed annually)



Notes:

1. TE (annual) is the ex-ante estimated TE for the return over the next month multiplied by $\sqrt{12}$.
2. The standardized return (annual) is the result of dividing realized monthly active return by the corresponding ex-ante TE estimate and multiplying by $\sqrt{12}$.
3. The solid line is equivalent to the efficient TE control for the active strategy evaluated. The slope of this line is calculated using equation (9).
4. The broken line is a 95% tile boundary of the standardized return (annual) when following normal distribution.
5. The gray line is the regression line when the standardized return is regressed to the estimated TE.

6. Conclusion

In this paper, efficient TE management in an active strategy was shown and the framework for evaluating active strategy using time series data of ex-ante TE estimates introduced. Under this concept, it was understood that, using the time series of ex-ante TE (risk) estimates of the active strategy in question, richer information which is never captured only by exploiting realized returns series[11], could be obtained.

Who can most benefit from this information are the investors entrusting their investment to fund managers. What these investors should do first and foremost is to view the total risk (TE) level taken by the fund manager over the periods and the transition in chart form, and to monitor the manager's moral hazard. In doing so, analyzing cross-sectional risk allocation towards each risk factor or evaluating dynamic TE control over the periods (see **Figure 1**), we are then able to shed even more light on each manager's characteristics from various angles.

The quantitative evaluation framework shown in this paper merely necessitates the acquisition of a small amount of additional data to clarify the traits of the active strategy. Complex knowledge is not required here. This means that the problem of asymmetric information between the investor and the fund manager can be greatly alleviated.

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References

(in Japanese)

Manabu Yano (2001) "Investment Risk Management through Risk Budgeting", *Security Analysts Journal*, Volume 39, No. 4, pp 47-64.

Motonari Kurasawa (1988), *Introduction to Price Theory* (2nd edition), Nippon-Hyoron-Sha.

Sadayuki Horie (2001) "Current State and Significance of Risk Budgeting," *Security Analysts Journal,*

[11] *In 'one period ahead' decision-making, an ex-ante TE (or risk) estimate is equivalent to the decision-making content and a realized return is equivalent to its ex-post result. By grasping the relation as a combination of these variables (or the ratio of these variables collectively called 'standardized return') in a multi-period setting, the fund manager's risk measurement capability (see subsection 5.1) and risk management capability (see subsection 5.5) can be evaluated in the original sense of MPT.*

Volume 39, No. 4, pp. 29-46.
(in English)

Grinold, R.C. and R. N. Kahn (1995), *Active Portfolio Management*, McGraw-Hill.