

SPECIAL FEATURE – Asset Allocation

Interest Rate Risk and Strategic Asset Allocation of Pension Funds

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Table of Contents

1. Introduction
2. Efficiency of the Bond Market Portfolio
3. Duration-Based Bond Pricing Model
4. Application to ALM
5. Implications for Practical Discussion
6. Conclusion
 - A. Derivation of Optimal Allocation of Stocks and Duration
 - B. Separation of the Optimization Problem
 - C. Additional Idiosyncratic Risk

Generally, strategic asset allocation determines the allocation of investment funds by asset class such as stocks and bonds; however, this is inadequate for bonds. Firstly because risk and return of bonds can be adjusted by duration which cannot be determined just by the allocation proportion, since most part of return of bonds can be captured by duration, and secondly, it is difficult to justify the use of bond market indices with predetermined duration as benchmarks.

Therefore, bond duration must be specified as well as bond allocation; this article presents a simple model of the issue of a pension fund's ALM. I consider the ALM issue because bonds can be positioned to hedge interest volatility risk with respect to liabilities. I then explain the behavior of the optimal portfolio and examine the impact of growth in liabilities due to a rise in wages and risks that cannot be completely explained by duration.

However, although the desired bond duration is obtained, how to make an investment that passively realizes a certain duration remains an issue. It must be different from 'indexing,' which aims to minimize tracking errors with the target set at certain indices.

1. Introduction

On a long-term basis, strategic asset allocation is a tool to enjoy the expected premiums paid for systematic risks, and is concretely expressed as a benchmark (portfolio). Enjoyment of premiums may seem to be a passive approach; however, a proactive approach is made as benchmarks, usually some kind of index, provide a return profile desirable for one's own investment purposes.

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Therefore, whether benchmarks are adequate is becoming increasingly important; however, there are serious reservations about bond investing (and benchmarks). For stocks, holding a market portfolio is supported under most general circumstances (assumption) based on the separation theorem. On the other hand, in the case of bonds, holding a market portfolio lacks sufficient foundation.

This is because bonds, unlike stocks, mature. With maturity, the positioning of safe and risk assets changes depending on the period of investments. A certain clientele effect is expected, and the availability of the optimal market portfolio that works for every investor comes into question. Nevertheless, many institutional investors hold a market (or a similar) portfolio¹.

In corporate pension management, investment decisions have been made focusing on only the return profile of assets. This is because interest in changes in the pension asset amount was believed to be sufficient as fluctuations in pension liabilities were not treated explicitly. However, an accounting standard for retirement benefits was introduced in FY2000 in Japan, and the difference between pension assets and liabilities is now reflected in corporate balance sheets; fluctuations in liabilities can no longer be ignored. For corporate management in general, the importance of matching debt fluctuation to investment returns is undoubted. Similarly, ALM provides a decision-making framework for pension fund management.

As the framework changes, so does the positioning of each asset class, and the change is especially significant in bonds. Liabilities as investment targets vary according to various factors, especially interest volatility risk, which has a profound impact on price fluctuation, which is also easy to measure. Bonds play a prominent role in ALM as this asset class is mostly affected by interest volatility risk.

Considering the ALM issue, risks and expected returns are controlled by the investment ratio for stocks and other assets, while a different view is possible for bonds because most of the bond return can be captured by the duration and largely adjusted without incurring much individual security risk. Thus, we can determine duration separately from the decision on the investment ratio. While investment in bonds based on a bond market portfolio is questionable from an optimality point of view, investments in indices with predetermined risk/return characteristics may not always be necessary.

This article explains how interest rate risk exposure (duration) is determined in pension fund strategic asset allocation, returning to the issue of bond pricing, by examining what the best benchmark for bonds is. In textbooks on bonds, the concept of duration or convexity is explained, but the relationship between this risk concept and a bond's return is rarely clarified. This article addresses this basic concept to show that alternative risk management, not based on the investment ratio, is important for bond investing².

The next chapter examines holding a bond market portfolio. Chapter 3 investigates bond return models using duration. Chapter 4 examines the ALM issue, explicitly solves optimal duration, and studies the behavior of the solution. Chapter 5 examines more realistic problems regarding current benchmarks and the effect of inflation. Chapter 6 concludes.

2. Effectiveness of the Bond Market Portfolio

First, I explain certain reservations about existing bond indices. Because of a market composed of investors with different maturity, a flexible supply of securities that are difficult to estimate, and additionally, characteristics of the duration of indices, using bond market indices as a benchmark for a normative portfolio is difficult to justify.

2.1 Equilibrium Price and Market Portfolio

With the security pricing model in the market, a representative investor is assumed to elicit a solution where demand and supply matches when the supply of securities is fixed. CAPM assumes this shape precisely and, as a result, indicates whether the market portfolio is effective, and each security (calculation of expected returns) can be priced based on covariance with the market portfolio. However, in the case of the bond market, this assumption is difficult to make.

This is partly because bonds mature. If the investment period is different, investors tend to hold mature bonds that match the

¹ For corporate pension fund management, NOMURA-BPI is used as the de-facto standard benchmark for bonds.

² Good examples are a series of articles by Leibowitz and related discussion published in the late 80s. See Leibowitz [11].

investment period. Bonds are free from risk as long as they are held until maturity³, and valuation of the individual bond varies depending on the holding period of the investor. Therefore, in the bond market, we cannot assume a representative investor to elicit fluctuation of prices from his/her behavior.

Thus, regarding a model to elicit the equilibrium price, discussion about investors who have different investing periods and their wealth is necessary; however, it is difficult to derive an equilibrium solution from matching demand and supply.

This is because the supply of securities changes flexibly, a second factor bringing the market portfolio into question. Bonds can be issued more easily than stocks. For example, if demand concentrates on a certain maturity, and the price of the bonds that become due on maturity rises (and the interest rate goes down), fund raisers issue the bonds accordingly. The bond market portfolio is the result of this behavior. Here, the investor's/issuer's behavior is determined by the assessment of undervaluation/overvaluation of interest rates, unlike stocks when they are interested in a wide range of diversified investment, and it is difficult to elicit the relationship with the bond market portfolio.

Adjustment of floating stocks by adjusting the cross-holdings of stable stockholders is recently being hotly debated regarding stock market indices. This is an effort to assess the amount of securities supplied with additional accuracy. How can the bond market be captured from this perspective?

In the bond market, investors who purchase securities upon issuance and retain them until redemption comprise quite a large proportion. For this reason, 'floating' bonds are limited to certain government bonds, for example. Considering the appropriateness of the benchmark, there is an opinion in which certain indices should be reluctantly recognized as the average for active management rather than an effective portfolio. However, as it is impossible to assess the balance of active investment, it is difficult to take this stand in the case of bonds. Issues posed by divergence between current bond market indices and the portfolio that expresses the real amount of bonds supplied are in fact more serious than the issue of stock indices, which are composed on the assumption that all outstanding stocks are available for investment.

2.2 Inverse Correlation between Duration and Interest Rates

Granito [7] termed the negative correlation shown between bond market indices duration and the level of interest rates (shorter when the interest rate is high and longer when low) 'index bias'. If an investor adjusts duration based on a bond's expected returns, under normal risk preference, duration becomes longer when the interest rate goes up and shorter when it goes down. From this point of view, he pointed out that index bias confounds the investor's expectations.

Index bias can also be observed empirically; issuer activities could be another contributing factor. In a low-interest rate climate, it is natural that financing by securities with longer redemption period increases. If an issuer's behavior is convincing, 'index bias' will also be generated structurally in the future. Therefore, it seems unreasonable to use market indices, where interest rate level and duration show inverse correlation, as investment criteria to represent a representative investor's risk preference.

3. Duration-Based Bond Pricing Model

If the canonicity of bond market indices as benchmarks is highly questionable, we have to explore some alternative benchmark for bond investing. One possibility is a characteristic that can capture the greater part of bond risk - duration.

In bond pricing, arguments are often based on a bond's price or yield; however, to construct a portfolio incorporating bonds, a bond's rate of return should be more appropriate. The rate of return can clarify the relationship with a stock portfolio, thus enabling the effective use of the results of the accumulated portfolio theory. As the impact of the interest rate on bonds can be captured by duration, if we can successfully relate the expected rate of return and duration, we should be able to elicit a more convenient pricing formula. This chapter explores a method of pricing based on duration⁴.

³ In reality, they are credit and the coupon's reinvestment risks, which are ignored in discussion focusing on the impact of bond maturity. In brief, a discount government bond is assumed.

⁴ For this and the next chapter, see Yonezawa = Omori [15].

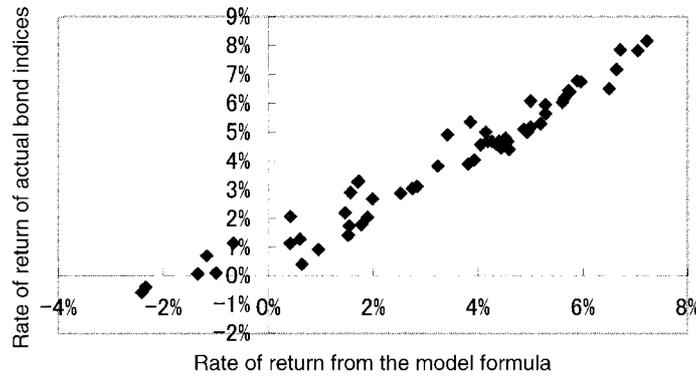
3.1 Relation Between Rate of Yield and Profitability

Supposing bond duration D is longer than period n , assuming the investment or valuation period to be the n -period, the approximate formula⁵ for return \tilde{R}_B is, according to Babcock [2], Leibowitz [11], and Campbell et. al [5]:

$$\tilde{R}_B \approx Y + \left(1 - \frac{D}{n}\right) \Delta \tilde{Y} \quad (1)$$

Here, Y and $\Delta \tilde{Y}$ represent the yield of the said bond and the difference between the current yield and the yield after n terms, respectively. Since $\Delta \tilde{Y}$ is currently a random variable, \tilde{R}_B is also a random variable. The impact of changes in the yield on the rate of return is $1 - \frac{D}{n}$, which is 0 or immunization if the investment period is equal to the duration.

Figure 1 Bond Return Model



Note: Nomura-BPI's 1-year rate of return from January 1999-November 2001 was used. The interest rate is the monthly average 7-year yield estimated from the government bond price. The model's explanatory power (R2) was 0.95, and the standard deviation of the difference in returns was 0.60%

This relationship was confirmed using ex-post index data (NOMURA-BPI) and obtained the results shown in Figure 1. The difference in returns is approximately 1% and seems to provide sufficient explanatory power as a rate of return model.

The analyzed period is used as the unit term: $n = 1$, and Formula (1) is translated into:

$$\begin{aligned} \tilde{R}_B &\approx E[\tilde{R}_B] + (1 - D) (\Delta \tilde{Y} - E[\Delta \tilde{Y}]) \\ &\approx E[\tilde{R}_B] + (D - 1) \tilde{\epsilon} \end{aligned} \quad (2)$$

($E[\cdot]$ represents expected value). Here, $E[\tilde{R}_B] = Y + (1 - D)E[\tilde{Y}]$ is the bond's expected rate of return, $\tilde{\epsilon}$ is the factor

⁵ This approximate expression is verified using a discount bond with maturity D :

$$\begin{aligned} \tilde{R}_B &= ((1 + Y + \Delta \tilde{Y})^{-(D-n)} (1 + Y)^D)^{\frac{1}{n}} - 1 \\ &= (1 + Y + \Delta \tilde{Y})^{1 - \frac{D}{n}} (1 + Y)^{\frac{D}{n}} - 1 \\ &\approx \left(1 - \frac{D}{n}\right) (Y + \Delta \tilde{Y}) + \frac{D}{n} Y = Y + \left(1 - \frac{D}{n}\right) \Delta \tilde{Y} \end{aligned}$$

Therefore, we can obtain the same formula as (1).

representing the short-term prediction error with a minus factor for the next term (zero on average). I put a minus here as the bond's price declines due to an increase in the short-term interest rate. I also assumed that the prediction errors of changes in the yield are approximated by the prediction errors in the short-term interest rate.

3.2 Expected Return

In the case of the bond portfolio, unlike stocks, the separation theorem does not function, and the market portfolio, which is good for all investors to hold, does not exist. Therefore, pricing based on the market portfolio, such as CAPM, is impossible⁶.

On the other hand, a return model that factors in changes in the interest rate has been obtained. As this factor broadly affects all bonds, we can apply APT to obtain:

$$E[\tilde{R}_B] = r + (D - 1) \delta \quad (3)$$

Here, r represents the short-term interest rate and δ the market price of risk. Value δ can be either positive or negative, depending on the investor's preference and the supply of securities; however, a positive figure is generally assumed. Assign this formula again to Formula (2):

$$\begin{aligned} \tilde{R}_B &= r + (D - 1) (\delta + \tilde{\epsilon}) \\ &= r + (D - 1) \tilde{\delta} \end{aligned} \quad (4)$$

(here, $E[\tilde{\delta}] = \delta$). From the next chapter, this formula will play a fundamental role. From now, $(D - 1)$ is sometimes expressed as D to avoid making the formulae too cumbersome.

4. Application to ALM

Now that a rate-of-return model has been obtained for bonds, let us examine strategic asset allocation for pension funds. By solving the ALM issue, we can discuss the desired asset allocation and optimal duration at the same time. In this regard, bond investing benchmarks should be seen as desirable interest rate risk exposure, i.e. duration. I first discuss liabilities as a management target and the ALM framework, then offer a solution to the ALM issue.

4.1 Fluctuation in Liabilities and Investment Targets

In the case of defined-benefit pensions, paying off a benefit must be covered by the investment returns of premiums and the relevant accumulated fund. There are several ways to calculate liabilities; here, we consider PBO (projected benefit obligation) in accordance with new standards of accounting. The amount is reflected in the corporate balance sheets based on the mark-to-market concept. PBO values consider prospective pay rises and are suitable for considering continuous investments over the long term.

The commonly used PBO formula can be summarized as follows, allowing for some inarticulacy:

$$\begin{aligned} \text{PBO} &= \sum_{i \in \text{Employees in active services}} \sum_{t=1}^{\text{Until official retirement age}} \text{Decrement rate}(t) \times G_i(t) \\ &\quad \times \frac{\text{Years of services}_i}{\text{Years of services}_{i+t}} \times (1 + \text{Discount rate})^{-t} \\ &\quad + \sum_{i \in \text{Retired employee}} \text{Present value of the benefit}_i \end{aligned}$$

⁶ However, as described later, when value is added by a single interest rate factor, the market portfolio may be a risk factor as a matter of course. For reference, Elton-Gruber [6] elicited the equilibrium formula of $E[\tilde{H}_i] = r + (E[\tilde{H}_m] - r)D_i/D_m$, assuming the holding of a market portfolio, but do not explain the reasons behind it. Here, \tilde{H}_m represents the rate of return of the bond market portfolio, D_m its duration, \tilde{H}_i the rate of return of bond i , and D_i its duration.

Here, i , t , and $G_i(t)$ represent an individual, time point of decrement (retirement), and the amount of allowance determined at separation (value at the time point of separation from service), respectively. Regarding generations in active service, the weighted average of the amount of benefits deemed as currently accrued based on the decrement ratio at each time point over the future of eligible employees is added. The present value of the allowances paid to retired employees is added to this to calculate PBO.

Considering changes in PBO for one year, we can first expect an increase due to prolonged service years (service cost), an increase due to decrease in the discount rate for one year's worth (interest cost), and decrease due to payment (cash out). However, the payment is already factored in the calculation, and the increase due to prolonged service years should be covered by the contribution. Therefore, the target of asset management is an increase through the rate of discount = interest rate.

Unexpected changes are next examined. Of the factors composing liabilities, the curve of the amount of allowance against years of service, the rate of separation, and personnel organization vary, largely depending on the individual situation of the company. It is also difficult to discuss the relationship with the asset market⁷; therefore, eliminating these fluctuations from this article, fluctuations in the discount rate remain. If the amount of allowance or other factors is fixed, since the structure is similar to that of the fixed coupon bond, the value of liabilities as the present value fluctuates according to discount rate fluctuation risk. The value of liabilities declines as the discount rate increases and rises as it decreases. In addition, the range of the value of liabilities, or volatility of rate of return, accords with the duration. Thus, the model in the previous chapter can be used as a liabilities fluctuation model.

The ALM concept depends on how the relation between pensions and the parent organization is seen. There is a lot of controversy over strategies, ranging from the 100% stock strategy that takes the pension compensatory payment scheme into account to maximize the put value, where the parent organization and the pension scheme are considered as a unit, and the strategy to maximize the tax arbitrage effect where the parent body issues bonds to raise proceeds for stock investment, and the pension funds are fully invested in bonds, to a strategy that even considers the parent body's business cycle⁸.

However, I focus on surplus in this argument. When the parent body and the pension scheme are considered as a unit, the reserve policy, taxes, original positioning of pension liabilities, and other considerations increase, and the issue becomes more complicated. It is beneficial to start discussion from a simple setting on the relation between the stock market and interest rate risks and asset allocation. In addition, regarding the decision-making framework, I adopted mean variance surplus optimization along the lines of studies by Sharpe = Tint [13] and other precedents, in preference to the ease of using the analytic approach.

4.2 Basic ALM Model

I now solve the optimization problem to explore optimal asset allocation and duration, First, I limit assumptions to the minimum, and add some extensions in the next chapter.

The duration of corporate pension liability, starts from Formula (5). Future benefit payments are as predetermined by pension scheme planning hereafter, which is beyond the scope of ALM direct operation. Two asset classes, stocks and bonds, can be made in bonds with any duration. Using Formula (4), returns on stocks, bonds, and liabilities for one time period can be modeled, respectively:

$$\tilde{R}_S = r + H_s \quad (6)$$

$$\tilde{R}_B = r + D\tilde{\delta} \quad (7)$$

$$\tilde{R}_L = r + \beta\tilde{\delta} \quad (8)$$

Here, D and β represent the duration of bonds and liabilities minus 1, respectively. r is the short-term interest rate, and \tilde{H}_S the fluctuation in stock return, represented as average H_S , and variance σ_s^2 , respectively. $\tilde{\delta}$ is the 'interest rate factor' introduced in the previous section, which is average δ or variance σ^2 . Covariance between the two random variables is set as $\text{Cov}(\tilde{H}_s, \tilde{\delta}) = \alpha\sigma^2$. α represents the interest rate sensitivity of stock return, which is equal to stock duration minus 1.

⁷ An increase in wages resulting from inflation or similar factors is considered in the next chapter.

⁸ Logue = Rader [12] introduced a broad range of arguments.

Suppose the inverse number of (assets/liabilities) is $K(\leq 1)$ ⁹, the portfolio is $x = (\text{stocks, bonds, and liabilities})' = (x_1, 1 - x_1, -K)$ and the measure of risk aversion is λ , then the optimization problem is:

$$\text{maximize}_{x_1, D} \nu'x - \frac{\lambda}{2} x'Qx, \quad (9)$$

$$\text{Where, } \nu = \begin{pmatrix} r + H_s \\ r + D\delta \\ r + \beta\delta \end{pmatrix} \quad Q = \begin{pmatrix} \sigma_s^2 & \alpha D\sigma^2 & \alpha\beta\sigma^2 \\ \alpha D\sigma^2 & D^2\sigma^2 & \beta D\sigma^2 \\ \alpha\beta\sigma^2 & \beta D\sigma^2 & \beta^2\sigma^2 \end{pmatrix}$$

Pension funds adjust asset allocation and bond duration to maximize risk adjusted surplus return. Thus far, the optimization problem of the portfolio including bonds has been discussed as an issue of developing a basic portfolio, but solely as the portfolio optimization of two asset classes, including stocks and bonds with a specific duration (or 3, including safe assets). Leibowitz [11] investigates about optimal portfolio duration; however, he does not touch on the relation between a bond's expected return and duration. However, Formula (9) is characterized by the inclusion of duration control from the risk/return perspective of the entire portfolio.

By solving Formula (9), the optimal percentage of stocks and bond duration (optimal D value + 1) is as follows (see Appendix for solution):

$$x_1^* = \frac{H_s - \alpha\delta}{\lambda(\sigma_s^2 - \alpha^2\sigma^2)} \quad (10)$$

$$D^* = \frac{\lambda\sigma^2(\sigma_s^2 - \alpha^2\sigma^2)\beta K - \alpha\sigma^2 H_s + \sigma_s^2\delta}{\sigma^2(\lambda(\sigma_s^2 - \alpha^2\sigma^2) - H_s + \alpha\delta)} + 1 \quad (11)$$

Here, the duration of the entire portfolio (interest sensitivity) can be expressed as follows¹⁰:

$$\begin{aligned} D_p &= D^*(1 - x_1^*) + (\alpha + 1)x_1^* \\ &= \beta K + 1 + \frac{\delta}{\lambda\sigma^2} \end{aligned} \quad (12)$$

From these results, we can see the following:

- The duration of liabilities bears no relation to the percentage of stocks. The percentage of stocks is determined by the attractiveness of the stocks based on their own expected return/risk after deducting the expected return/risk attributable to the interest rate factor.
- The duration of the entire portfolio is also determined independently of the stocks. Duration is determined to carry risks in response to the attractiveness of the interest rate factor, based on the duration of liabilities or immunization.
- Bond duration is adjusted so that the entire portfolio reaches the status described in the previous item.
- When risk aversion is significant, or λ is large, the percentage of stocks and bond duration end in 0 and $\beta K + 1$, respectively. This is immunization.
- The effect of diminishing surplus and K becoming closer to 1 means that debt duration is extended. βK are the risk factor, not independently.
- If changes in stocks are independent of interest rate fluctuation ($\delta = 0$), bond duration is:

⁹ Like this, ALM that aims to control surplus cannot deal with a case in which $K > 1$ (shortage of reserves). As a shortage of reserves is now seen frequently, care is needed. When it is not necessary to consider liabilities at all, replace $K = 0$ before optimization.

¹⁰ If surplus return is translated into exposure to stocks and the interest rate factor before optimization, Formula (10) and (12) are obtained more easily (see Appendix B, clause 5-3). However, this formulation was adopted to clearly show relevance to the asset allocation issue.

$$D^* = \frac{\lambda \sigma_s^2 \sigma^2 \beta K + \sigma_s^2 \delta}{\lambda \sigma_s^2 \sigma^2 - \sigma^2 H_s} + 1 \quad (13)$$

For both the interest rate factor and stocks, the greater the risk premium, the longer the duration.

- In addition, if there is no premium in the interest rate factor ($\delta = 0$), risk (σ_s) is eliminated from the duration formula. In other words, the size of interest rate risk has no effect on decision-making. Duration is then adjusted in such a way that the entire portfolio is immunized.

We can establish an image using concrete figures. The results of calculating the optimal percentage of stocks and duration under various conditions are shown in Table 1.

When stocks' expected returns increase (cases 1, 2), although the percentage of stocks rises, we can see that the duration of bonds is adjusted and that of the entire portfolio is unchanged. We can then see that the percentage of stocks is unchanged when the duration of liabilities changes (cases 2-4). Without an interest rate premium (Case 5), duration of the entire portfolio is $K + 1$ (immunization). From cases 6-7, we can see that the duration of bonds shortens in proportion to the percentage of stocks when the duration of stocks is long¹¹. This is because stocks assume the role of bonds as hedge assets. However, in this case, duration of the entire portfolio also stays constant.

Table 1 Optimized Portfolio

	H_s	δ	α	β	x_1^*	D^*	D_p
Case1	2%	0.1%	1	15	9.5%	16.37	15.00
Case2	5%	0.1%	1	15	24.6%	19.23	15.00
Case3	5%	0.1%	1	10	24.6%	13.93	11.00
Case4	5%	0.1%	1	18	24.6%	22.41	17.40
Case5	5%	0.0%	1	15	25.1%	16.68	13.00
Case6	2%	0.1%	16	15	5.6%	14.88	15.00
Case7	7%	0.1%	16	15	75.0%	9.00	15.00
Case8	5%	0.08%	1	15	24.7%	18.72	14.60

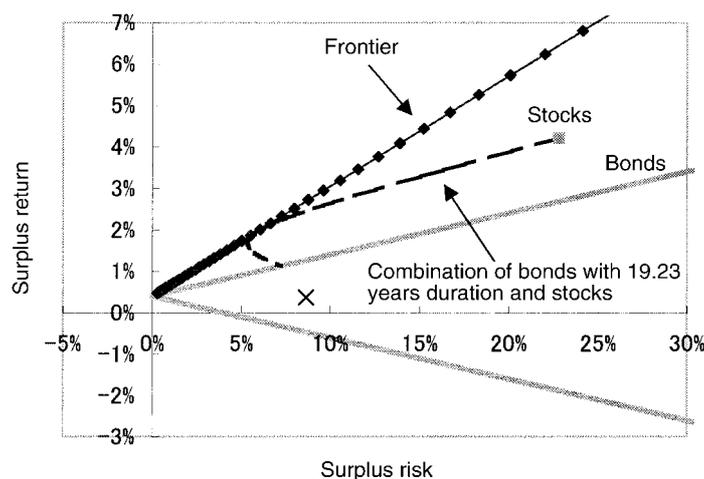
Note: Other parameters are: $s = 20\%$, $\sigma = 1\%$, $K = 0.8$ and $\lambda = 5$.

The frontier of surplus risk/return in Case 2 is shown in Figure 2. Bond characteristics are shown as the broken line as they vary depending on duration. The point where the curving line representing the combination of the bond with a 19.23-year duration and stocks meets the frontier provides the optimal portfolio for Case 2. The extreme left of the frontier represents the immunization strategy.

In addition, both optimal duration model formulae (11) and (12) are expressed as 'risk factor sensitivity function + 1' because the sensitivity of returns for one period is modeled as 'duration -1;' therefore, for return during n periods, n periods should be adjusted accordingly. Furthermore, when obligations fluctuate due to a risk factor other than the interest rate factor, if the fluctuation is caused by the same factor and any investment asset that can adjust to this exposure without restraint is available, we can obtain the same result for optimal exposure.

¹¹ Though stock duration observed from market data is small, if a dividend discount model is used for stock valuation instead of market price, stock duration generally becomes quite long (Bostock et al. [4]).

Figure 2 Efficient Frontier for Surplus



Note: Risk/return figures by asset amount = 1. The same parameters as in Case 2 of Table 1 were used, and $r = 2\%$. x represents the optimal portfolio of 5.5-year fixed bonds and stocks.

5. Implications for Practical Discussion

Based on the basic models above, several topics about bond investment are explored. First, why the duration of the actual bond benchmark is significantly shorter than that suggested by the ALM model. Secondly, inflation, which is always cited as evidence against immunization strategy, the impact of the increase rate of wages due to improved labor productivity and foreign assets. Lastly, the impact of respective idiosyncratic risks attaching to bonds and liabilities on strategic asset allocation, which cannot be completely captured by the interest rate factor.

Table 2 Recent Government Bond Market: Overview

	Issuing Cycle	Amount per issue	Duration	Transaction cost
10-year govt bonds	Every month	Approx JPY1.7 trillion	Approx 9 years	1bp
20-year govt bonds	Every 6 months	Approx JPY0.5 trillion	Approx 16 years	1.5bps
30-year govt bonds	2-3 times a year	Approx JPY0.3 trillion	Approx 21 years	2bps

Note: Amounts issued and other public offering data in late 2001. Transaction cost = difference between offer and bid on a simple interest basis; figures shown are based on securities companies, not actual transaction data, for indication only.

5.1 What Shortens Duration

Currently, the benchmark duration commonly used by pension funds is approximately 5.5 years and significantly shorter than the solution of the previous model. The optimal portfolio on a 5.5-year fixed bond duration basis is also shown in Figure 2 and we can see that it falls significantly below the efficient frontier. Therefore, I examined transaction cost and the low volatility in the liabilities valuation rate of interest as the relevant factors here; however, these factors could not provide sufficient explanation, and short durations could not be justified in terms of ALM¹².

5.1.1 Transaction Cost of Ultra Long-Term Bonds

To construct a bond portfolio with a duration of over 10 years, investments in 20- or 30-year government bonds are required in

¹² Under current low-interest rate conditions where the distribution of future bonds or liabilities returns deviates to the left (no significant growth), it may appear that duration should be shortened. Of course, this opinion is worth considering from a tactical viewpoint for individual funds; however, argument here about strategic asset allocation ignores this.

reality. High transaction costs due to low liquidity are often mentioned as a drawback to investments in ultra long-term bonds. An overview of the recent government bond market is shown in Table 2. Ultra long-term government bonds (20 or 30 years) are certainly smaller in terms of amount issued and incur higher transaction costs compared to long-term government bonds (10 years). How significant is this impact?

According to Table 2, the duration of a 30-year bond is 21 years, and the transaction cost is approximately 42bps in terms of price difference. If a bond is purchased and sold within an investment period of one year, this amount of cost (loss) is incurred in total. As seen here, the cost is indicated as yield and is almost in proportion to duration; therefore, it is possible to incorporate the cost as it is into the models in this article. In the case of ultra long-term bonds, assume δ is lower by 0.02%. The non-linearity of cost (in the case of long-term bonds, δ only declines 0.01%) should naturally be considered as well as its absolute value; however, we can regard them as proportional for an approximation.

The result of δ decreased by 0.02% is shown in Case 8 of Table 1. Compared with Case 2, although bond duration is shortened by approximately 0.5 years, it is still nearly 20 years and produces no major change. In addition, here cost is incorporated into the model so that it can be amortized in one year. However, the investment period of the actual bond portfolio by security is longer, and the effect of transaction cost should be smaller. Therefore, the transaction estimated expressly does not have sufficient impact to significantly shorten duration.

However, I made test calculations on the premise that any transaction volume is possible. For example, transactions involving hundreds of millions of yen per ultra long-term bond issue (approximately JPY100 billion by fund size) may be insignificant. What happens if a number of pension funds move toward purchase to prolong the duration?

As shown in Table 2, the supply of ultra long-term bonds is significantly less than that of long-term bonds; therefore, prices may increase due to tightness of supply. However, in this case, investors make investments in ultra long-term bonds, placing them as something close to risk-free assets; an interest rate decline cannot categorically be given as a disadvantage. In the meantime, the government and other issuers who make ultra long term investments (in real assets) should increase bond supply. The balance between investment and financing in the ultra long-term fund market will be more accurately reflected in the interest rate; therefore, tightness in ultra long-term bonds due to pension fund behavior may only be a transitional matter.

5.1.2 Volatility of the Reference Interest Rate for Debt Valuation

In fact, the interest rate applied to discount cash flow for debt valuation is not the market interest rate. The opinion brief of the Business Accounting Council states that “the interest rate can be determined, taking fluctuations in the bond yield during a certain period,” and, according to JICPA’s Practical Guidelines of Accounting for Retirement Benefits, “it can be adjusted considering the fluctuations in the yield for a certain period in the past.” In this way, a certain amount of discretion of the parent body is allowed for the interest rate applied. In practice, a 5-year average yield of government bonds is commonly used as a guidepost, ranging broadly from 1.5-3.5% as affairs stand.

Based on the average of the yield, it is equal to decreased volatility in the interest rate. If interest rate fluctuation is simply independent by the year, volatility falls to slightly less than half ($1/\sqrt{5}$) by considering the 5-year average. To express this in the debt model, adjusting the interest rate premium separately, we only need to decrease duration to slightly less than half, correspondingly. This is because, for example, liabilities valuation does not increase even though the interest rate declines and actual debt cash flow value increases, as the discount rate applied reflects the decrease in the interest rate only partially.

As a solution to the ALM issue, bond duration will be shortened significantly. Since liabilities are unlikely to change, the result will be similar to the outcome before introduction of the accounting standard for retirement benefits.

However, care is needed to affirm the short duration of the asset portfolio based on low volatility in the reference interest rate because, in line with the model’s shape, the predicted value of the future reference interest rate has changed even though interest volatility declines.

In the case of interest rate decline, as the market interest rate is lower than the reference interest rate, the reference interest rate is likely to decline in the future without convincing prospects that the interest rate will increase again. In other words, a downward trend is created in the reference interest rate. In this case, the unrealized loss incurred is highly likely to be tangible at some stage (as liabilities are greater than valuation in reality); therefore, its soundness is questionable. The true market value of liabilities is unknown, and appraisal value has a margin of error regardless of the accounting system applied. However, the expected margin of error must be 0.

5.2 Increase in Liabilities Due to Rise in Wages

So far, I have discussed returns from bonds and liabilities focusing on changes resulting from interest rate fluctuation. Leibowitz [8, 9, 10] offers a theory focusing on changes in short-term surplus, particularly from a similar point of view. However, there have always been opposing arguments on the effect of a long-term increase in wages (Ambachtsheer [1], Bookstaber = Gold [3] etc.).

Ambachtsheer [1] focuses on the growth in debt resulting from inflation and improved productivity, pointing out that although investments in stocks (sometimes including investments in real estate and unlisted stocks) involve a huge risk over the short term, they are attractive in decreasing the risk of assets falling below liabilities over the long term.

Here, I examine the question of vital interest, an increase in wages. The relation between inflation and other factors and the amount of benefit is determined by the pension scheme or the outcome of labor negotiations, and a definitive, one-on-one relationship may not necessarily exist. However for example, a substantial decrease in pension benefits due to inflation may not be left completely derelict.

First, with respect to expected growth due to inflation and improved productivity, the problem is how it is incorporated. For bonds, assuming 'interest rate = expected inflation + real economic growth,' we can see that the returns from inflation and other factors are already incorporated in cash flow. For stocks, too, if an investor anticipates and evaluates future cash flow in real terms, expected inflation and other factors are already incorporated.

On the other hand, this may be not appropriate for liabilities. Currently, PBO calculation in Japan is often made based on the ongoing wage table¹³. In other words, liabilities grow in accordance with the increase in cash flow along with inflation, outpacing growth by the interest rate.

However, this does not have a significant impact on the optimal portfolio. It is the risk premium of the stocks and the interest rate factor that have an impact despite high growth in liabilities. It is as if the short-term interest rate has declined to depress the overall rate of returns.

There is a great change when the unexpected impact of inflation or real growth is considered. First, in the case of bonds in general, cash flow is fixed against inflation so there is no sensitivity to changes in inflation, etc. The bond portfolio is somewhat sensitive due to refinancing; however, sensitivity should not be great. As stocks are valued in real terms, they show growth in response to unexpected inflation or similar changes¹⁴.

In the case of liabilities, the value of sensitivity to changes in the expected increase rate in wages due to inflation or similar factors should be similar to that of duration with contrary signs. For example, if the amount of benefit is linked to inflation, a 1% increase in the inflation rate is equal to a 10% increase or more in terms of the amount payable after 10 years. The longer the term is the greater the impact. Inflation or other factors may have little impact on the defined benefits of retired employees; however, it should be close to the duration, which is the average maturity for all liabilities.

With a high correlation between the interest rate and the increase rate of wages considered, the effective duration of liabilities will be shorter. For example, if the interest rate increases, it is as if liabilities have declined, but the rate of inflation increases at the same time, and thus, future benefit cash flow grows by the same amount.

If we apply this to model Formula (9), covariance between stocks and liabilities increases, and the effective interest rate volatility risk attaching to liabilities declines due to the 'wage growth rate factor.' Therefore, stocks become more appealing as a hedge asset against the risk of an increase in wages, while bond duration is diminished. Both effects are unfavorable for bonds.

5.3 Foreign Assets

I have classified investment assets into two classes: stocks and bonds. I now make a distinction between domestic/foreign securities, classifying assets into these categories to approximate the actual portfolio construction.

The following conditions were established to present a better picture for discussion. First, currencies are basically fully hedged

¹³ JICPA's Practical Guidelines of Accounting for Retirement Benefits states that "the expected fluctuation in the future wage level shall not be included in the forward wage increase ratio, unless reasonably estimable in a convincing way;" currently, an increase in wages along with inflation or other factors is not incorporated in most cases. However, under the economic circumstances where inflation is experienced regularly, this can be interpreted as "should be incorporated."

¹⁴ There is a lot of controversy over the hedging ability of stocks with respect to inflation, etc.; fundamentally, estimation based on ultra long-term data and a structural analysis of liabilities/macro-economy is required.

and managed independently of other assets; in other words, hedging is removed or forward exchange contracts (or forward agreements) are bought when currency exposure is necessary. Suppose changes in foreign stocks can be described by a single foreign market factor other than Japan. Foreign bonds are dependent on global currency fluctuation; however, here we consider the 'global currency basket' to consolidate fluctuation in bond returns into a single factor, which represents interest-rate fluctuations. Under normal conditions, risk/return outlook should be provided for each currency. However, here I explore the fluctuation factor of the currency basket.

The return fluctuation model for foreign stocks is determined by a foreign stock market factor itself. On the other hand, suppose that risks and returns on foreign bonds and currencies are proportional to the relevant interest rate/currency factor exposure. In other words, we can apply a similar modeling to Formula (7) for domestic bond returns and Formula (8) for liabilities returns.

I do not consider the short-selling restriction. In this way, the duration of foreign bonds can be adjusted by combining long- and short-term bonds. For foreign currencies, future trading is permitted without restriction; the factor exposure can be adjusted arbitrarily in the case of domestic/foreign bonds and currencies, regardless of the investment ratio. Regarding the optimization issue, of the three asset classes added to Formula (9), normally two match the investment ratio in exposure (domestic and foreign stocks). Another three asset classes (domestic and foreign bonds and currencies) are characterized so that their risk exposure can be determined regardless of the investment ratio.

A complex explanation with more algebraic symbols is required for a solution to the optimization issue; here, I explain the results of the arguments thus far about cases in which there is no correlation between domestic/foreign stocks and changes in domestic/foreign interest rate fluctuations. As the correlation between changes in stocks and interest rates varies depending on the timing or data cycle, assuming non-correlation is feasible. On the other hand, as some similarity is found among domestic/foreign bonds, foreign currencies and fluctuations in the interest rate in each currency, covariance should also be discussed.

Under these circumstances, the optimization issue can be separated into domestic/foreign stocks and interest-related matters (domestic and foreign bonds, currencies and liabilities) as shown in the Appendix B. First, the investment ratio of domestic/foreign stocks should be treated in a similar way to an ordinary asset allocation issue. In other words, allocation is determined based on the attractiveness of each stock; the ratio of respective risk premium and risk deducted by the portion can be explained by correlation between domestic and foreign stocks.

The availability of premiums in foreign bonds and currencies is also important for interest rate-related assets. For example, if the interest rate risk premium (δ of the model in Formula (7)) of foreign bonds is greater than that of domestic bonds even in terms of correlation with the domestic interest rate, we can extend duration by holding foreign bonds. For foreign currencies, too, if higher returns are expected than from the domestic short-term interest rate, exposure will be taken after considering the relevant risk. In any case, the duration of the domestic bonds will be adjusted from the status before the introduction of foreign assets by correlation between interest rates. However, as interest rates do not correlate with each other completely, both provide poorer hedging ability against liabilities interest volatility risk than domestic bonds; there is a lot of controversy over the availability of premiums that sufficiently exceed the premiums of domestic bonds. Therefore, the risk exposure of foreign bonds and currencies should remain small if taken, unlike domestic bonds.

Considering the above as a problem of individual funds, constraints should usually be added. First, as short selling is difficult in the case of bonds, the span of the adjustable range of duration is narrowed; the issue cannot be separated as association with the stock investment. As the ratio of investment in foreign assets usually creates an upper limit for foreign currencies, foreign currency exposure is also associated with asset allocation here. These demands can be easily solved by numerical analysis; however, in this study, a simple setting was employed to make inferences about strategic allocation in general.

Table 3 Impact of Idiosyncratic Risk

	\sqrt{e}	\sqrt{f}	x_1^*	D^*	D_p
Case1	2%	0%	25.3%	19.41	15.00
Case2	5%	0%	29.0%	20.31	15.00
Case3	10%	0%	39.7%	23.55	15.00
Case4	2%	10%	25.3%	19.41	15.00

Note: other parameters are $r = 0.1\%$, $H_s = 5.0\%$, $\beta = 1$, $\lambda = 15$, $\sigma_s = 20\%$, $\sigma_l = 1$ K = 0.8 and $\lambda = 5$.

5.4 Idiosyncratic Risk

In fact, neither liabilities nor assets can be completely explained by the interest rate, inflation, or other factors. They have own idiosyncratic risks. These idiosyncratic risks have not been discussed, and I now verify their impact.

As idiosyncratic risks have no relation to factor fluctuations, when they are considered in Formula (9), the covariance matrix is changed to:

$$Q = \begin{pmatrix} \sigma_s^2 & \alpha D \sigma^2 & \alpha \beta \sigma^2 \\ \alpha D \sigma^2 & D^2 \sigma^2 + e & \beta D \sigma^2 \\ \alpha \beta \sigma^2 & \beta D \sigma^2 & \beta^2 \sigma^2 + f \end{pmatrix} \quad (14)$$

e and f (0) represent the idiosyncratic risk of bonds and liabilities, respectively¹⁵. We can obtain a solution to the analysis through this change (see Appendix C); however, I only show the behavior of the optimal solution as the expression is too complex.

The results are shown in Table 3. By comparing cases 1-3 and Case 2 in Table 1, we can see that the larger the idiosyncratic risk of bond e , the greater the percentage of stocks. This must be because stocks enhance the relative appeal as bond risk, which is unrelated to returns or hedging demand, increases. However, the risk unattributable to bond duration accounts for only a few percentage points and should not have a great influence.

By comparing Case 1 and Case 4, we can see that idiosyncratic liabilities risk exercises no effect. By analogy with Formula (5), liabilities should carry a significant idiosyncratic risk due to changes in plan beneficiaries, staff turnover, or other factors that are difficult to control, which need not be considered for asset allocation. However, this is because this risk cannot be hedged by asset management, which still needs to be controlled in some other way.

6. Conclusion

In general, strategic asset allocation determines the allocation of investment funds by asset class such as stocks and bonds. However, this is inadequate for bonds, firstly because it allows latitude in risk and return adjustment by duration, which cannot be determined by the allocation proportion alone, as the better part of return from bonds can be captured by the duration, and secondly, because it is difficult to justify the use of bond market indices with a predetermined duration as benchmarks.

Therefore, in the case of bonds, duration must be determined at the same time as the allocation ratio. This article adds the bond duration dimension to the commonly used mean variance analysis to construct a strategic asset allocation model, which was applied to the ALM issue. I examined the ALM issue as bonds are useful as a hedging tool against liabilities interest rate fluctuation risk. I then explained the behavior of the optimal portfolio and examined the short duration issue of the bond benchmarks currently in use and liabilities growth due to wage increase as well as the effect of idiosyncratic risks.

Thus, I used duration as the reference indicator. To apply them in practical investment, there are still a number of problems in

¹⁵ Like this, if we model idiosyncratic bond risk as a fixed number, as duration becomes longer, bond risk attributable to the interest rate factor increases, which is favorable for risk management. As this article emphasizes optimal duration, this bias is not desirable. To solve this, we could model bond risk as $D^2 \sigma^2 (1 + e)$ for example. However, this article sets idiosyncratic risk as a fixed number because of the small bias and because long-term bonds carry a great risk in the latter model. Practical judgment should be based on empirical analysis.

carrying out actual operation. This is partially due to there being no such asset as duration.

Even if the adequate duration for strategic allocation is obtained, it is still necessary to clarify the issue of passive management for realization. For example, a portfolio that specifies the duration to minimize other risks should be wholly composed of discount bonds that reach maturity within the same year, but this conclusion is unacceptable. Ideal passive management means that the portfolio can passively accept interest rate fluctuation. This method of management must be different from 'indexing,' which aims to minimize tracking errors with the target set at certain indices.

If more detailed information is available regarding liabilities, we can discuss further. There are no investment criteria other than duration because of investment target information, and liabilities are consolidated into duration. To study more suitable passive management, we can apply an extensive liabilities information approach. In concrete terms, examples include sensitivity to inflation or wage increase and expansion of the liabilities model incorporating a higher degree of duration. We can also consider expanding the liabilities return model to credit and other non-interest rate risks.

In addition, the pension fund reserve currently often falls below liabilities ($K > 1$). Is normal risk aversion setting adequate to make decisions in an area where the surplus is negative? To obtain a solution that is convincing in practice, further strategies in utility function and the framework itself are required.

Here, I examined strategic asset allocation assuming passive management, but the valuation of active management also obviously changes. Above all, failing to set benchmarks for bonds after defining the relation between risk and returns may only increase risk in active management.

Appendix A Derivation of Optimal Allocation of Stocks and Duration

If bond duration is directly used as a decision variable, it may be difficult to relate it to the asset allocation issue. It is easier to address it as the quadratic program issue regarding general asset allocation by introducing long- or short-term bonds. As duration is specified by the ratio of long- and short-term bonds, without a short-selling restriction, the ratio between stocks and bonds and duration can be optimized at the same time.

The rate of return for one year of long- and short-term bonds can be modeled as follows. L and S represent the duration of long- and short-term bonds deducted by 1:

$$r + L\tilde{\delta}, r + S\tilde{\delta}, \quad (15)$$

Suppose Portfolio x is (stocks, long-term bonds, short-term bonds, and debt) $= (x_1, x_2, x_3, -K)$, the optimization problem is:

$$\text{maximize } \mu'x - \frac{\lambda}{2}x'Qx, \quad (16)$$

$$\text{subject to } \sum_{i=1}^3 x_i = 1,$$

The expected return vector and covariance matrix is:

$$\mu = \begin{pmatrix} r + H_s \\ r + L\tilde{\delta} \\ r + S\tilde{\delta} \\ r + \beta\tilde{\delta} \end{pmatrix}, \quad (17)$$

$$Q = \begin{pmatrix} \sigma_s^2 & \alpha L\sigma^2 & \alpha S\sigma^2 & \alpha\beta\sigma^2 \\ \alpha L\sigma^2 & L^2\sigma^2 & LS\sigma^2 & L\beta\sigma^2 \\ \alpha S\sigma^2 & LS\sigma^2 & S^2\sigma^2 & S\beta\sigma^2 \\ \alpha\beta\sigma^2 & L\beta\sigma^2 & S\beta\sigma^2 & \beta^2\sigma^2 \end{pmatrix} \quad (18)$$

By expressing the limiting condition and the percentage of reserve data as:

$$\begin{aligned}
x &= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -K \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= t + T\bar{x}, \tag{19}
\end{aligned}$$

and translating the objective function as:

$$\begin{aligned}
\mu'(t + T\bar{x}) - \frac{\lambda}{2}(t + T\bar{x})'Q(t + T\bar{x}) \\
= (\mu - \lambda Qt)'T\bar{x} - \frac{\lambda}{2}\bar{x}'T'QT\bar{x} + \text{Constant}, \tag{20}
\end{aligned}$$

Under these conditions, this can be transformed to:

$$\begin{aligned}
T'(\mu - \lambda Qt) &= \hat{\mu} \\
&= \begin{pmatrix} H_s - S\delta - \lambda(\alpha - S)(S - \beta K)\sigma^2 \\ (L - S)(\delta - \lambda(S - \beta K)\sigma^2) \end{pmatrix}, \tag{21}
\end{aligned}$$

$$\begin{aligned}
T'QT &= \hat{Q} \\
&= \begin{pmatrix} \sigma_s^2 + S^2\sigma^2 - 2\alpha S\sigma^2 & (L - S)(\alpha - S)\sigma^2 \\ (L - S)(\alpha - S)\sigma^2 & (L - S)^2\sigma^2 \end{pmatrix} \tag{22}
\end{aligned}$$

The optimal solution can then be obtained as¹⁶:

$$\begin{aligned}
\hat{x}^* &= \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \frac{1}{\lambda}\hat{Q}^{-1}\hat{\mu} \\
&= \begin{pmatrix} \frac{H_s - \alpha\delta}{\lambda(\sigma_s^2 - \alpha^2\sigma^2)} \\ \frac{(S - \alpha)\sigma^2 H_s + (\sigma_s^2 - \alpha S\sigma^2)\delta}{\lambda(L - S)\sigma^2(\sigma_s^2 - \alpha^2\sigma^2)} + \frac{\beta K - S}{L - S} \end{pmatrix} \tag{23}
\end{aligned}$$

Here, duration is:

$$\begin{aligned}
D^* &= \frac{Lx_2^* + S(1 - x_1^* - x_2^*)}{1 - x_1^*} + 1 \\
&= \frac{\lambda\sigma^2(\sigma_s^2 - \alpha^2\sigma^2)\beta K - \alpha\sigma^2 H_s + \sigma_s^2\delta}{\sigma^2(\lambda(\sigma_s^2 - \alpha^2\sigma^2) - H_s + \alpha\delta)} + 1 \tag{24}
\end{aligned}$$

Since the attributes of long-/short-term bonds (L and S) and the short-term interest rate are not included in optimal duration (24), we can verify that the subject formulation is equivalent to independent control over duration. Suppose there is no correlation between stocks and the interest rate, or, $\rho = 0$, it can be simplified to:

$$\hat{x}^* = \begin{pmatrix} \frac{H_s}{\lambda\sigma_s^2} \\ \frac{S\sigma^2 H_s + \sigma_s^2(\delta + \lambda(\beta K - S)\sigma^2)}{\lambda(L - S)\sigma^2\sigma_s^2} \end{pmatrix} \tag{25}$$

$$D^* = \frac{\sigma_s^2(\lambda\sigma_s^2\beta K + \delta)}{\sigma^2(\lambda\sigma_s^2 - H_s)} + 1 \tag{26}$$

Appendix B Separation of the Optimization Problem

Suppose the portfolio vector to be $y =$ (domestic and foreign stocks, domestic and foreign bonds, foreign currency, and liabilities),

¹⁶ Here, $\sigma_s^2 = \alpha^2\sigma^2$ only when the variance of the stock return is explained by the interest rate factor; however, this is physically impossible. Therefore, this solution exists.

then segmented into domestic/foreign stocks and another four interest rate-related asset classes (including liabilities) and expressed as $y=(y_1, y_2)$. Here, suppose each y factor represents exposure to the risk factor, which can be adjusted without restraint, when there is no short-selling restriction. Returns generated by short-term interest are ignored hereafter since it is fixed, relying on the net volume of investment $((1-K)r)$, and has no relevance to the risk factor.

First, these covariance matrices can be expressed as the following block diagonal matrix by Q_1 with 2×2 dimensions, relating to domestic/foreign stocks and Q_2 with 4×4 dimensions, relating to interest rate-related assets, assuming that the stocks have no correlation with the interest rate:

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \quad (27)$$

In addition, due to the restriction in which liabilities is K -fold of the assets:

$$(0, 0, 0, 0, 0, 1)y = -K \quad (28)$$

Therefore, this can be divided into y_1 and y_2 without difficulty. By dividing the expected return in the same manner into $v=(v_1, v_2)$, the optimization problem is:

$$\begin{aligned} \text{maximize } & v'y - \frac{\lambda}{2} y' Q y \\ & = \left(v_1' y_1 - \frac{\lambda}{2} y_1' Q_1 y_1 \right) \\ & \quad + \left(v_2' y_2 - \frac{\lambda}{2} y_2' Q_2 y_2 \right), \end{aligned} \quad (29)$$

$$\text{subject to } (0, 0, 0, 1)y_2 = -K$$

We can see that it is possible to dissect the problem into parts: one part relating to two types of stock and the other part relating to interest rate-related assets.

The second term of Formula (29), as the first variable (domestic bonds) perfectly correlates with the fourth variable (liabilities), will be transformed again. y_2 can be further divided into \hat{y}_2 which represents three asset classes and liabilities. The expected return and the covariance can then be divided into three- and one-dimensional parts as follows:

$$v_2 = \begin{pmatrix} \hat{v}_2 \\ \beta \delta \end{pmatrix}, \quad Q_2 = \begin{pmatrix} q_{11} & q_{12} \\ q_{12}' & \beta^2 \sigma^2 \end{pmatrix}$$

q_{11} represents the covariance matrix of domestic/foreign bonds and foreign currency; as they have their own respective fluctuation components, there is an inverse matrix. The objective function for interest rate-related assets is transformed to:

$$\begin{aligned} & v_2' y_2 - \frac{\lambda}{2} y_2' Q_2 y_2 \\ & = (\hat{v}_2 + \lambda K q_{12})' \hat{y}_2 - \frac{\lambda}{2} \hat{y}_2' q_{11} \hat{y}_2 + \text{Constant} \end{aligned} \quad (30)$$

Therefore, the optimal solution is:

$$y_1^* = \frac{1}{\lambda} Q_1^{-1} v_1, \quad \hat{y}_2^* = \frac{1}{\lambda} q_{11}^{-1} (\hat{v}_2 + \lambda K q_{12}) \quad (31)$$

Then, by relating the result to Formula (9), which does not cover foreign assets and currencies, since $v_1 = H_S$, $Q_1 = \sigma_s^2$, $\hat{v}_2 = \delta$, $q_{11} = \sigma^2$, $q_{12} = \sigma^2$, we can obtain $\beta = 0$ in Formula (10) for the optimal allocation of stock investment, and Formula (12) for the optimal portfolio duration.

Appendix C Additional Idiosyncratic Risk

In the same way, I here introduce long- and short-term bonds to reduce the issue regarding the asset allocation of two asset classes.

The covariance matrix in Formula (18) is changed into:

$$Q = \begin{pmatrix} \sigma_s^2 & \alpha L \sigma^2 & \alpha S \sigma^2 & \alpha \beta \sigma^2 \\ \alpha L \sigma^2 & L^2 \sigma^2 + e & LS \sigma^2 + e & L \beta \sigma^2 \\ \alpha S \sigma^2 & LS \sigma^2 + e & S^2 \sigma^2 + e & S \beta \sigma^2 \\ \alpha \beta \sigma^2 & L \beta \sigma^2 & S \beta \sigma^2 & \beta^2 \sigma^2 + f \end{pmatrix} \quad (32)$$

As diversified investment effect between long- and short-term bonds is avoided, they both have the same idiosyncratic regulation. By transforming the formula in the same manner as described in the previous section, the expected return vector and the covariance matrix for this two-asset class issue is:

$$\hat{\mu} = \begin{pmatrix} H_s - S\delta - \lambda(\alpha - S)(S - \beta K)\sigma^2 + \lambda e \\ (L - S)(\delta - \lambda(S - \beta K)\sigma^2) \end{pmatrix}, \quad (33)$$

$$\hat{Q} = \begin{pmatrix} \sigma_s^2 + S^2 \sigma^2 - 2\alpha S \sigma^2 + e & (L - S)(\alpha - S)\sigma^2 \\ (L - S)(\alpha - S)\sigma^2 & (L - S)^2 \sigma^2 \end{pmatrix} \quad (34)$$

Then, we only need to calculate $\hat{x}^* = \frac{1}{\lambda} \hat{Q}^{-1} \hat{\mu}$. Interestingly, the idiosyncratic risk of liabilities f has disappeared.

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